15-150

Principles of Functional Programming

Slides for Lecture 14

Regular Expressions

March 11, 2025

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Lessons:

- Regular Expressions
- Regular Languages
- Matcher
- Correctness
  - Proof-Directed Debugging
  - Termination
  - Soundness and Completeness

Language Hierarchy

Class of Languages	Recognizers	Applications
Unrestricted	Turing Machines	General Computation
Context-Sensitive	Linear-bounded automata	Some simple type-checking
Context-Free	Nondeterministic automata with one stack	Syntax checking
Regular	Finite Automata	Tokenization

An example: Excursions from home

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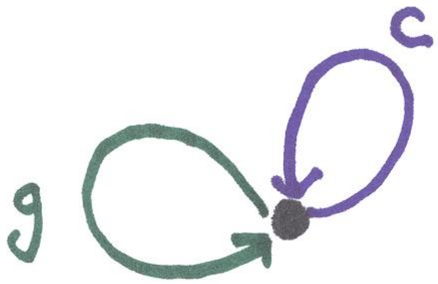
"**c**" means "go to CMU, then go home"

An example: Excursions from home



"**c**" means "go to CMU, then go home"

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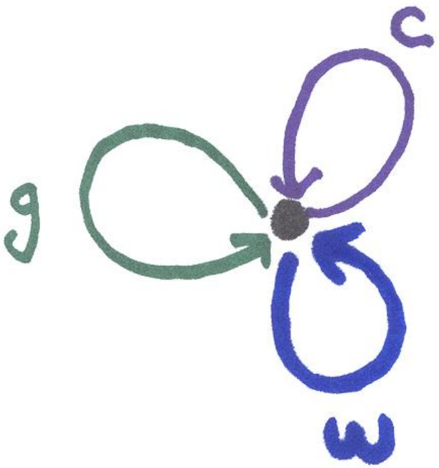

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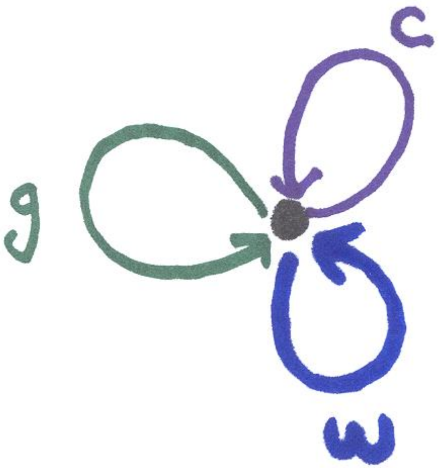


"**c**" means "go to CMU, then go home"

"**g**" means "get groceries, then go home"

"***w***" means "go for a walk, then home"

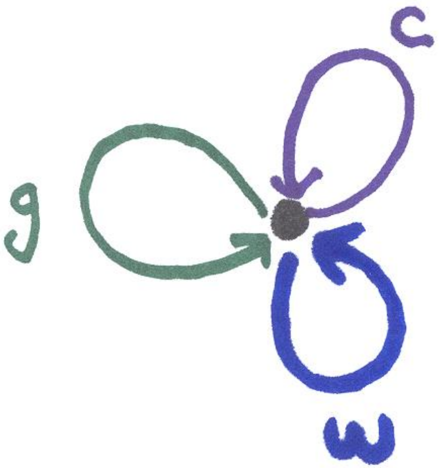
Description of excursions in a given week:

c (go to CMU once) **cc** (go to CMU twice) **ccc** (go to CMU 3 times)

c* (go to CMU zero or more times)

cgc (go to CMU, then get groceries, then go to CMU)

An example: Excursions from home



"**c**" means "go to CMU, then go home"

"**g**" means "get groceries, then go home"

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<u>Description of excursions in a given week:</u>

g + *w* (get groceries **OR** go for a walk)

(g + *w*)* (zero or more times do one of the following: get groceries **OR** go for a walk)

(g + *w*)*c (zero or more times do one of the following: get groceries **OR** go for a walk; *after that* go to CMU once)

Notation and Definitions

Σ is an *alphabet* of *characters*. (nonempty, finite)

For example, $\Sigma = \{a, b\}$.

(Using SML , **`#"a" : char`**.)

Σ* means the set of all finite-length strings over alphabet **Σ**, i.e., with characters in **Σ**.

For example, aabba is in {a,b}*.

(Using SML , **`"aabba" : string`**.)

ε is the *empty string*, containing no characters.

ε is in **Σ***.

(Using SML , **`"" : string`**.)

Notation and Definitions

A *language* over Σ is a subset of Σ^*.

(In other words, a language is a set of finite-length strings with characters in Σ. A language may contain infinitely many strings.)

We are here interested in a particular class of languages called ***regular languages***. The languages may have infinite size, but we will describe them via a finite representation called ***regular expressions***, much like in the excursion example.

Regular Expressions

Assume we have been given some alphabet Σ.

A *regular expression* over Σ is any of the following:

Regular Expressions

Assume we have been given some alphabet Σ.

A *regular expression* over Σ is any of the following:

a for every character $\mathbf{a} \in \Sigma$,

set symbol meaning “is in”

(don’t confuse with the empty string ε)

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$\mathbf{r_1 + r_2}$	with $\mathbf{r_1}$ and $\mathbf{r_2}$ regular expressions (called *alternation*),

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$\mathbf{r_1 r_2}$	with $\mathbf{r_1}$ and $\mathbf{r_2}$ regular expressions (called *concatenation*),
r*	with **r** a regular expression (called *Kleene star*).

Regular Expressions

Assume we have been given some alphabet Σ.

A *regular expression* over Σ is any of the following:

(And use parentheses as needed.)

$\mathbf{r_1 + r_2}$	with $\mathbf{r_1}$ and $\mathbf{r_2}$ regular expressions (called *alternation*),
$\mathbf{r_1 r_2}$	with $\mathbf{r_1}$ and $\mathbf{r_2}$ regular expressions (called *concatenation*),
$\mathbf{r^*}$	with $\mathbf{r}$ a regular expression (called *Kleene star*).

Regular Languages

Given regular expression **r** we define language **L(r)**:

L(a) = {a} (singleton set) for every character **a** $\in \Sigma$,

L(0) = { } (the empty language, no strings),

L(1) = {ε} (the language consisting of the empty string),

$\mathbf{L(r_1 + r_2) = \{ s \mid s \in L(r_1)}$ or $\mathbf{s \in L(r_2) \}}$ (not exclusive),

$\mathbf{L(r_1 r_2) = \{ s_1 s_2 \mid s_1 \in L(r_1)}$ and $\mathbf{s_2 \in L(r_2) \}}$,

$\mathbf{L(r^*) = \{ s \mid s = s_1 s_2 \cdots s_n}$, some $\mathbf{n} \geq 0$, with each $\mathbf{s_i \in L(r) \}}$

(here we mean **s = ε** when n=0).

So: $\boldsymbol{\varepsilon} \in \mathbf{L(r^*)}$ for all regular expressions **r**.

Regular Languages

Let **Σ** be a given alphabet and **L** a subset of **Σ***.

We say that language **L** is *regular* if **L = L(r)** for some regular expression **r**.

(Fact: The class of regular languages over **Σ** is the minimal class containing the empty set and all singleton subsets of **Σ**, and that is closed under union, concatenation, and Kleene star.)

(The class is also closed under complement:

L is regular iff **Σ* ** L is regular.)

Examples (assume $\Sigma = \{a, b\}$)

L(a) = {a} (singleton set consisting of the string **a**)

L(aa) = {aa} (singleton set consisting of the string **aa**)

L((a + b)*) = Σ* (all finite-length strings with **a**s and **b**s)

L((a + b)*aa(a + b)*) = all strings in **Σ*** containing at least two consecutive **a**s.

L((a + 1)(b + ba)*) = **?????**

Examples (assume $\Sigma = \{a, b\}$)

L(a) = {a} (singleton set consisting of the string **a**)

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L((a + b)*aa(a + b)*) = all strings in **Σ*** containing at least two consecutive **a**s.

L((a + 1)(b + ba)*) = all strings in **Σ*** that do *not* contain two consecutive **a**s.

Examples (assume $\Sigma = \{a, b\}$)

Comment: Different regular expressions can give rise to the same regular language.

For instance:

L(ab + b*ab)
= L((1 + b*)ab)
= L((1 + bb*)ab)
= L(b*ab)
= L(b*ab + 0)
= all strings in **Σ*** consisting of zero or more **b**s followed by **ab** (and nothing thereafter).

Examples (assume $\Sigma = \{a, b\}$)

Comment: Different regular expressions can give rise to the same regular language.

For instance:

> In particular, for any reg exp **r**:
>
> **L(r*) = L(1 + rr*)**

L(ab + b*ab)

= L((1 + b*)ab)

= L((1 + bb*)ab)

= L(b*ab)

= L(b*ab + 0)

= all strings in **Σ*** consisting of zero or more **b**s followed by **ab** (and nothing thereafter).

An Acceptor

We would like to implement a function that decides whether a given string **s** is in the language **L(r)** of a given regular expression **r**.

```
(* accept : regexp -> string -> bool
   REQUIRES: true (may change this later).
   ENSURES: (accept r s) returns true if s ∈ L(r);
            (accept r s) returns false, otherwise.
*)
```

Think of `accept` as a simple parser/compiler.

(Still need to define the `regexp` type.)

Matching

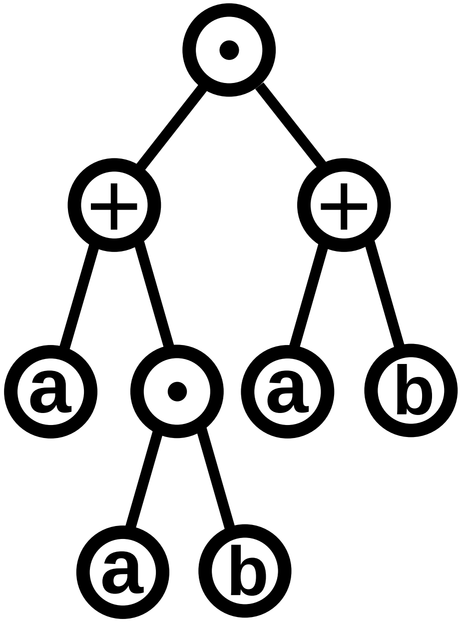
Suppose **r = (a + ab)(a + b).**

Then **L(r) = {aa, ab, aba, abb}.**

How does the acceptor recognize that **aba ∈ L(r)** ?

By backtracking search.

View **r** as a tree.



Use up characters in **aba** matching tree operations determined by **r**.

Matching

(a + ab)(a + b)

a **ba**

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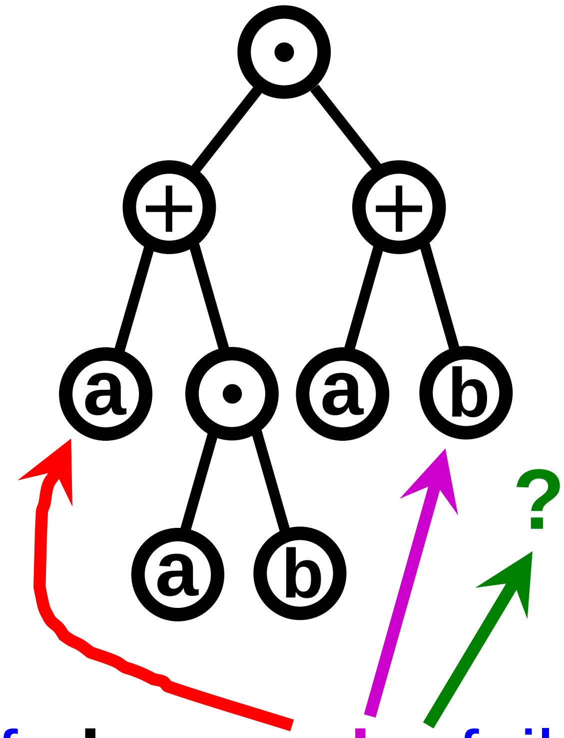

First split of **aba** as **a ba** fails on last character.

Matching

(a + ab)(a + b)

ab **a**

Suppose **r = (a + ab)(a + b).**

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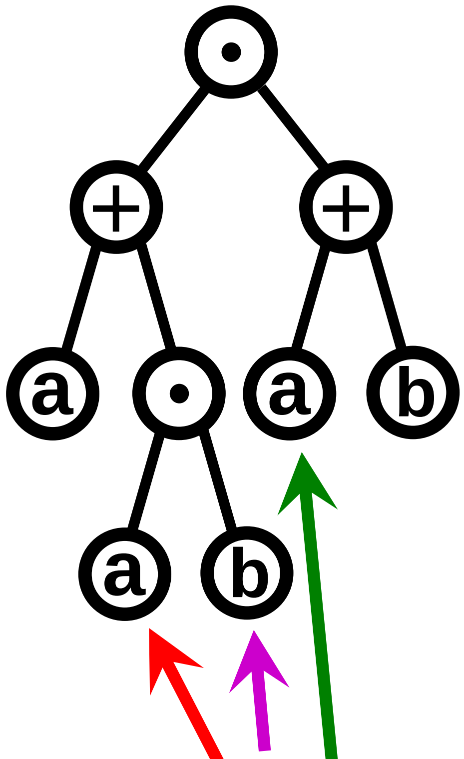

Second split of **aba** as **ab** **a** succeeds.

Matching

Suppose **r = (a + ab)(a + b).**

Then **L(r) = {aa, ab, aba, abb}.**

How does the acceptor recognize that **aba ∈ L(r)** ?

By backtracking search.

Tonight, do an evaluation trace on this example of the code we are about to write.

(Check yourself using today's lecture page.)

A Matcher

We will implement the backtracking search using a Boolean-specific continuation.

```
(* match : regexp -> char list ->
                          (char list -> bool) -> bool

   REQUIRES: k is total (aside: weaker condition
                        simplifies termination proof).

   ENSURES: (match r cs k) returns true if
                 cs can be split as cs≅p@s, with
                 p representing a string in L(r)
                 and k(s) evaluating to true;
             (match r cs k) returns false, otherwise.
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```

We use character lists instead of strings here for simplicity.
In discussions/proofs we sometimes treat them as identical.

Acceptor Based on Matcher Specs

```
(* match : regexp -> char list ->
                     (char list -> bool) -> bool
   REQUIRES: k is total.
   ENSURES: (match r cs k) ≅ true if
              cs ≅ p@s, with p ∈ L(r) & k(s) ≅ true;
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   accept : regexp -> string -> bool
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   fun accept r s =
         match r (String.explode s) List.null
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```

turns a string into a char list

```
fun accept r s =
      match r (String.explode s) List.null
```

`List.null : 'a list -> bool` decides whether a list is empty.

Implementation

We will define a datatype that mirrors the mathematical definition of regular expressions.

We will implement a matcher that mirrors the definition of a regular expression's language.

Implementation

```
datatype regexp =
            Char of char
          | Zero
          | One
          | Plus of regexp * regexp
          | Times of regexp * regexp
          | Star of regexp
```

Implementation

```
fun match
```

Implementation

```
fun match (Char a) cs k =
```

Implementation

```
fun match (Char a) cs k =
      (case cs of
            [] =>
          | c::cs' =>                                    )
```

Implementation

```
fun match (Char a) cs k =
      (case cs of
           [] => ?????
         | c::cs' =>                                  )
```

Recall:

`(match r cs k)` ≅ `true`

if `cs` ≅ `p@s`, with `p` ∈ `L(r)` & `k(s)` ≅ `true`

L(a) = {a}

Implementation

```
fun match (Char a) cs k =
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Implementation

```
fun match (Char a) cs k =
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```

Implementation

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fun match (Char a) cs k =
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        | c::cs' => (a=c) andalso (k cs'))
  | match Zero _ _ = ?????
```

Recall:

(match r cs k) ≅ true

if cs ≅ p@s, with p ∈ L(r) & k(s) ≅ true

L(0) = { }

Implementation

```
fun match (Char a) cs k =
      (case cs of
          [] => false
        | c::cs' => (a=c) andalso (k cs'))
  | match Zero _ _ = false
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  | match One cs k = ?????
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Recall:

(match r cs k) ≅ true

if cs ≅ p@s, with p ∈ L(r) & k(s) ≅ true

L(1) = {ε}

Implementation

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```

Implementation

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fun match (Char a) cs k =
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  | match Zero _ _ = false
  | match One cs k = k cs
  | match (Plus(r1,r2)) cs k =
```

Implementation

`(match r cs k)` ≅ `true`

if `cs` ≅ `p@s`, with `p` ∈ `L(r)` & `k(s)` ≅ `true`

$L(r_1 + r_2) = \{ s \mid s \in L(r_1) \text{ or } s \in L(r_2) \}$

```
| match (Plus(r₁,r₂)) cs k =
    (match r₁ cs k)
```

?????

Implementation

```
fun match (Char a) cs k =
      (case cs of
           [] => false
         | c::cs' => (a=c) andalso (k cs'))
  | match Zero _ _ = false
  | match One cs k = k cs
  | match (Plus(r1,r2)) cs k =
      (match r1 cs k) orelse (match r2 cs k)
```

Implementation

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fun match (Char a) cs k =
      (case cs of
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  | match (Plus(r1,r2)) cs k =
      (match r1 cs k) orelse (match r2 cs k)
  | match (Times(r1,r2)) cs k =
```

Implementation

`(match r cs k)` ≅ `true`

if `cs` ≅ `p@s`, with `p` ∈ `L(r)` & `k(s)` ≅ `true`

$L(r_1 r_2) = \{ s_1 s_2 \mid s_1 \in L(r_1) \text{ and } s_2 \in L(r_2) \}$

```
| match (Times(r1,r2)) cs k =
   match r1 cs
```

?????

Implementation

$$\texttt{(match r cs k)} \cong \texttt{true}$$

if $\texttt{cs} \cong \texttt{p@s}$, with $\texttt{p} \in \texttt{L(r)}$ & $\texttt{k(s)} \cong \texttt{true}$

$$L(r_1 r_2) = \{\, s_1 s_2 \mid s_1 \in L(r_1) \text{ and } s_2 \in L(r_2) \,\}$$

```
| match (Times(r1,r2)) cs k =
   match r1 cs (fn cs' =>  ?????      )
```

Implementation

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     match r1 cs (fn cs' => match r2 cs' k)
```

Implementation – Star clause

```
| match (Star r) cs k =
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Recall: **L(r*) = L(1 + rr*)**

We could make calls to previous clauses, but let's implement this equation directly.

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```

Recall: **$L(r^*) = L(1 + rr^*)$**

We could make calls to previous clauses, but let's implement this equation directly.

There is a potential bug.

```
| match (Star r) cs k =
   (k cs) orelse (match r cs (fn cs' => match (Star r) cs' k))
```

Proof-Directed Debugging

```
| match (Star r) cs k =
   (k cs) orelse (match r cs (fn cs' => match (Star r) cs' k))
```

Imagine trying to prove that **`(match (Star r) cs k)`** reduces to a value as part of some larger induction proof that **`match`** always *terminates* (returns a value) when given input satisfying the specs.

In the Induction Hypothesis we may assume that **`(match r cs k)`** reduces to a value whenever **`k`** is total. So we need to establish that **`(fn cs' => match (Star r) cs' k)`** is total. Now we are in a circular argument!

Proof-Directed Debugging

```
| match (Star r) cs k =
  (k cs) orelse (match r cs (fn cs' => match (Star r) cs' k))
```

A possible way out: We don't really need to establish that

```
(fn cs' => match (Star r) cs' k)
```

is total, merely that it returns values when called on suffixes **`cs'`** of the given **`cs`**. Maybe a second induction on **`cs`** will help.

If we could show that **`cs'`** is a *proper suffix* of **`cs`**, we could perhaps establish eventual termination.

Proof-Directed Debugging

```
                                              k'
| match (Star r) cs k =
  (k cs) orelse (match r cs (fn cs' => match (Star r) cs' k))
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If we could show that `cs'` is a *proper suffix* of `cs`, we could perhaps establish eventual termination.

ALAS, that need not be true:

```
match (Star One) [#"a"] List.null
```

will loop forever since `List.null [#"a"]` ≅ `false` and since `match One cs k'` will pass all of `cs` to `k'`.

Proof-Directed Debugging

```
                                                          k'
| match (Star r) cs k =
  (k cs) orelse (match r cs (fn cs' => match (Star r) cs' k))
```

This issue arises when the empty string is in **L(r)**.

If we could show that `cs'` is a *proper suffix* of `cs`, we could perhaps establish eventual termination.

ALAS, that need not be true:

```
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will loop forever since `List.null [#"a"] ≅ false` and since `match One cs k'` will pass all of `cs` to `k'`.

Two possible fixes to avoid infinite loops

1. Change the specs:
   - Require regular expressions to be in *standard form* (definition shortly).

2. Change the code:
   - Explicitly check that **`cs'`** is a proper suffix of **`cs`**.

Two possible fixes to avoid infinite loops

1. Change the specs

Definition:

A regular expression **r** is in *standard form* iff for any subexpression **Star(r')** of **r**, **L(r')** does *not* contain the empty string **ε**.

Fact: It is possible to convert any regular expression **r** into a regular expression **q** that is in standard form such that **L(r) = L(q)**.

Consequently, if we **REQUIRE** regular expressions to be in standard form we avoid infinite loops without losing any regular languages.

(Preprocess **r** into standard form, then call **match**.)

Two possible fixes to avoid infinite loops

2. Change the code

```
| match (Star r) cs k =
    k cs    orelse
    match r cs (fn cs' =>
                        properSuffix (cs',cs)
                              andalso
                        match (Star r) cs' k)
```

Two possible fixes to avoid infinite loops

2. <u>Change the code</u>

```
| match (Star r) cs k =
    k cs   orelse
    match r cs (fn cs' =>
                    properSuffix (cs',cs)
                          andalso
                    match (Star r) cs' k)
```

This is new.

The code checks that **cs'** is a proper suffix of **cs**.

Sketch of a Proof of Correctness

1. Prove Termination

Show that **`(match r cs k)`** returns a value for all arguments **`r`**, **`cs`**, **`k`** satisfying **`REQUIRES`** specs.
(This proof is surprisingly difficult. We assume it here.)

2. Prove Soundness and Completeness

Given termination, we can simplify the **`ENSURES`** specs in a convenient way, then perform structural induction.
(We will write out one of the recursive cases here.)

Soundness & Completeness, Assuming Termination

Here are the given **ENSURES** specs for **match**:

```
(match r cs k) ≅ true if cs ≅ p@s,
                         with p ∈ L(r) and k(s) ≅ true;
(match r cs k) ≅ false, otherwise.
```

Given termination, we can rephrase the specs as:

```
(match r cs k) ≅ true if and only if there exist p and s
      such that cs ≅ p@s, p ∈ L(r), and k(s) ≅ true.
```

That is the theorem we must prove.

The "if" part is sometimes called "completeness".

The "only if" part is sometimes called "soundness".

Theorem

For all values

r : regexp, **cs : char list**, **k : char list -> bool**, with **k** total,

$$\texttt{(match r cs k)} \cong \texttt{true}$$

if and only if

there exist **p** and **s** such that

cs $\cong$ **p@s**, **p** $\in$ **L(r)**, and **k(s)** $\cong$ **true**.

Theorem

For all values

r : regexp, **cs : char list**, **k : char list -> bool**, with **k** total,

(match r cs k) ≅ true

if and only if

there exist **p** and **s** such that

cs ≅ p@s, **p ∈ L(r)**, and **k(s) ≅ true**.

(We are assuming termination as a lemma.)

Theorem

For all values

`r : regexp`, `cs : char list`, `k : char list -> bool`, with `k` total,

`(match r cs k)` ≅ `true`

if and only if

there exist `p` and `s` such that

`cs` ≅ `p@s`, `p` ∈ L(`r`), and `k(s)` ≅ `true`.

(We are assuming termination as a lemma.)

Proof

By structural induction on `r`.

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Base Cases: `Zero`, `One`, `Char(a)` for every `a:char`.

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Plus$(\mathbf{r}_1,\mathbf{r}_2)$, **Times**$(\mathbf{r}_1,\mathbf{r}_2)$, **Star(r)**.

Theorem

For all values

r : regexp, **cs : char list**, **k : char list -> bool**, with **k** total,

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(We are assuming termination as a lemma.)

Proof

By structural induction on **r**.

Base Cases: **Zero**, **One**, **Char(a)** for every **a:char**.

Inductive Cases:

Plus(r_1,r_2), **Times**(r_1,r_2), **Star(r)**.

We will discuss only the **Plus** case here, as an example.

(See also today's online notes, including another proof technique.)

Inductive Case `r = Plus(r₁,r₂)`, for some r_1, r_2 :

IH: For $i=1,2$ and for all values **cs** & **k**, with **k** total,

(match r_i cs k) $\cong$ **true** iff there exist **p**&**s** such that **cs** $\cong$ **p@s**, **p** $\in$ L(r_i), & **k(s)** $\cong$ **true**.

NTS: For all values **cs** & **k**, with **k** total,

(match (Plus(r_1,r_2)) cs k) $\cong$ **true** iff there exist **p**&**s** such that **cs** $\cong$ **p@s**, **p** $\in$ L(**Plus(r_1,r_2)**), & **k(s)** $\cong$ **true**.

(We will prove the two parts of the "iff" separately.)

I. Suppose `(match (Plus(`r_1`,`r_2`)) cs k)` ≅ `true`.

NTS: There exist `p`&`s` such that `cs` ≅ `p@s`,

`p` ∈ L(`Plus(`r_1`,`r_2`)`), & `k(s)` ≅ `true`.

Showing:

`true`

[assumption] ≅ `(match (Plus(`r_1`,`r_2`)) cs k)`

[`Plus`] ≅ `(match` r_1 `cs k) orelse (match` r_2 `cs k)`

∴ One or both of the arguments to `orelse` must be `true`.

Let us suppose it is the first argument (proof similar for second).

So `(match` r_1 `cs k)` ≅ `true`.

By IH for r_1,

there exist `p`&`s` s.t. `cs` ≅ `p@s`, `p` ∈ L(r_1), & `k(s)` ≅ `true` .

Then also `p` ∈ L(`Plus(`r_1`,`r_2`)`), by language definition for `Plus`.

That finishes this part of the proof (soundness).

II. Suppose there exist **p**&**s** such that **cs** ≅ **p@s**, **p** ∈ L(**Plus(r**$_1$**,r**$_2$**)**), & **k(s)** ≅ **true**.

NTS: **(match (Plus(r**$_1$**,r**$_2$**)) cs k)** ≅ **true**.

Showing:

		(match (Plus(r$_1$**,r**$_2$**)) cs k)**
[**Plus**]	≅	**(match r**$_1$ **cs k) orelse (match r**$_2$ **cs k)**
[see below]	≅	**true**

By supposition, there exist **p**&**s** such that **cs** ≅ **p@s**, **p** ∈ L(**Plus(r**$_1$**,r**$_2$**)**), & **k(s)** ≅ **true**. By the language definition for **Plus**, **p** ∈ L(**r**$_1$) and/or **p** ∈ L(**r**$_2$).

If **p** ∈ L(**r**$_1$) , then **(match r**$_1$ **cs k)** ≅ **true** by IH for **r**$_1$.

Otherwise, **(match r**$_1$ **cs k)** ≅ **false** by termination, **p** ∈ L(**r**$_2$), and **(match r**$_2$ **cs k)** ≅ **true** by IH for **r**$_2$.

That finishes this part of the proof (completeness), and so the **Plus** case.

That is all.

Please have a good lab.

See you Thursday.

We will discuss another matcher,
inspired by staging and combinators.